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which is satisfied by $b = 2$; for then

$$c^2 = 4n^4 + 12n^3 + 17n^2 + 12n + 4 = (2n^2 + 3n + 2)^2,$$

and $z = 4(n^2 + n + 1)$. This value of z , with the assumed values, $x = n^2$, $y = (n + 1)^2$, satisfies all the proposed conditions.

$$xy + z = n^2(n + 1)^2 + 4(n^2 + n + 1) = (n^2 + n + 2)^2,$$

$$yz + x = 4(n + 1)^2(n^2 + n + 1) + n^2 = (2n^2 + 3n + 2)^2,$$

$$xz + y = 4n^2(n^2 + n + 1) + (n + 1)^2 = (2n^2 + n + 1)^2.$$

If $n = 1$, then $x = 1$, $y = 4$, $z = 12$.

If $n = 2$, then $x = 4$, $y = 9$, $z = 28$.

If $n = 3$, then $x = 9$, $y = 16$, $z = 52$.

And so on, indefinitely.

The values of x , y , z just found will also satisfy the conditions

$$xy + x + y = \square, \quad xz + x + z = \square, \quad \text{and} \quad yz + y + z = \square.$$

Also solved by ELIZABETH B. DAVIS and H. N. CARLETON.

237. Proposed by NORMAN ANNING, Chilliwack, B. C.

Prove that for three numbers x , y , z ,

$$9\Sigma(x - y)^4 = \Sigma(2x - y - z) = 2\square.$$

SOLUTION BY E. F. CANADAY, University of South Dakota.

This problem is evidently misprinted. If we write it

$$9\Sigma(x - y)^4 = \Sigma(2x - y - z)^4 = 2\square,$$

a solution is possible. To prove

$$9[(x - y)^4 + (y - z)^4 + (z - x)^4] = (2x - y - z)^4 + (2y - z - x)^4 + (2z - x - y)^4 = 2\square,$$

we put

$$(x - y) = a, \quad (y - z) = b, \quad \text{and} \quad (z - x) = -(a + b).$$

Then

$$\begin{aligned} 9[a^4 + b^4 + (-a - b)^4] &= (2a + b)^4 + (b - a)^4 + (-a - 2b)^4 = 9(2a^4 + 4a^3b + 6a^2b^2 \\ &\quad - 4ab^3 + 2b^4) = 16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4 + b^4 - 4ab^3 + 6a^2b^2 - 4a^3b + a^4 + a^4 \\ &\quad + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4 = 2[9(a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4)] = 18a^4 + 36a^3b \\ &\quad + 54a^2b^2 + 36ab^3 + 18b^4 = 2\square. \end{aligned}$$

$$2[3(a^2 + ab + b^2)]^2 = 2\square.$$

Also solved by the PROPOSER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence, Kansas.

REPLIES.

20. Some of our readers would like to have a simple account, without proofs, of just what has been accomplished toward the proof of the theorem that the equation $x^n + y^n = z^n$ is impossible in integers when $n > 2$.

Readers interested in the above question will be glad to learn that a better and more complete article than that contemplated as an answer to the question

is soon to appear in the *Annals of Mathematics*. The paper is being prepared by Professor L. E. Dickson, of the University of Chicago, and will be a somewhat extended account of the more important results in proper historical setting.

Reference may also be made to A. Fleck's six-page article on Fermat's last theorem in Auerbach and Rothe's *Taschenbuch für Mathematiker und Physiker*, 3. Jahrgang, 1913 (Teubner), pp. 103–109; Benno Lind's forty-three-page article, "Über das letzte Fermatsche Theorem" in *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Heft XXVI₂, pp. 21–65, 1910; and Lipke's four-page review of Lind's article (*Bull. Amer. Math. Soc.*, Vol. XVIII, pp. 194–198) in which references are given to the criticisms of Lind's work published in *Archiv der Mathematik und Physik*.

33. Under what conditions or to what extent is Mr. Iwerson's construction a useful or practical approximation to a true ellipse? What criterion can be given to measure definitely the degree of approximation?

Mr. Iwerson's approximate construction for an ellipse by ruler and compasses alone, having given the axes, was given in the November, 1916, issue of the *MONTHLY*, pp. 354, 355. The following corrections should be made: In the figure, B' should be B . In the last two lines, Ox should be OY , and Oy should be OX .

REPLY BY PAUL CAPRON, U. S. Naval Academy, Annapolis, Md.

In the figure above referred to, let $XO = OX' = a$, $YO = OY' = b$. Then $X'A = a$, $X'B = (a^2 - b^2)/\sqrt{a^2 + b^2}$. Let $k = AB$ and $l = NN' = NR$. Then $k = a - (a^2 - b^2)/\sqrt{a^2 + b^2}$, and $l = 2a - k = a + (a^2 - b^2)/\sqrt{a^2 + b^2}$. Since $X'N'P'$ and $N'NR$ are equilateral triangles,

$$RO = (l - k) \frac{\sqrt{3}}{2} = \frac{a^2 - b^2}{\sqrt{a^2 + b^2}} \cdot \sqrt{3}, \quad \text{and} \quad l - RO = a - \frac{a^2 - b^2}{\sqrt{a^2 + b^2}} (\sqrt{3} - 1).$$

In order that the arc drawn with R as center and l as radius (through N and N') may pass through Y , $l - RO$ must be equal to b . This is the case (if $b < a$) when and only when $a = b\sqrt{3}$.

Let $b/a = x$ (if the eccentricity is e , $e^2 + x^2 = 1$). The proportional error in the length of the minor axis is

$$E_1 = \frac{1}{b} [l - RO - b] = \frac{1-x}{x} \left[1 - \frac{1+x}{\sqrt{1+x^2}} (\sqrt{3}-1) \right].$$

The proportional errors in the radii of curvature at the ends of the axes are: at the end of the major axis,

$$E_2 = \frac{a}{b^2} \left(k - \frac{b^2}{a} \right) = \frac{a^2 - b^2}{b^2} \left(1 - \frac{a}{\sqrt{a^2 + b^2}} \right) = \frac{1-x^2}{x^2} \left(1 - \frac{1}{\sqrt{1+x^2}} \right);$$

at the end of the minor axis,

$$E_3 = \frac{b}{a^2} \left(l - \frac{a^2}{b} \right) = \frac{a-b}{a^2} \left[\frac{b(a+b)}{\sqrt{a^2+b^2}} - a \right] = (1-x) \left[\frac{x(1+x)}{\sqrt{1+x^2}} - 1 \right].$$

$$\frac{dE_1}{dx} = \left[(\sqrt{3}-1) \frac{1+3x^2}{(1+x^2)^{3/2}} - 1 \right], \quad \frac{dE_2}{dx} = \frac{1}{x^3} \left[\frac{2+3x^2-x^4}{(1+x^2)^{3/2}} - 2 \right],$$

$$\frac{dE_3}{dx} = \frac{1-3x^2-2x^4}{(1+x^2)^{3/2}} + 1.$$

Of the six variables, E_2 and dE_2/dx vanish for no real value of x between 0 and 1; the values, aside from 0 and 1, which cause the variables to vanish, are given by the following equations:

$$E_1 = 0; \quad (x - \frac{1}{3}\sqrt{3})(x - \sqrt{3}) = 0,$$

$$\frac{dE_1}{dx} = 0; \quad x^6 + 3(6\sqrt{3} - 11)x^4 + 3(4\sqrt{3} - 7)x^2 + (2\sqrt{3} - 3) = 0,$$

$$\frac{dE_2}{dx} = 0; \quad x^4 - 10x^2 - 7 = 0,$$

$$E_3 = 0; \quad x^4 + 2x^3 - 1 = 0,$$

$$\frac{dE_3}{dx} = 0; \quad 4x^6 + 11x^4 + 2x^2 - 9 = 0.$$

It is said that architects find it troublesome to draw a shapely oval opening; the ellipse is the most satisfactory curve in itself, but there is much labor involved in making offsets for several parallel curves in the design of the mouldings. With Mr. Iwerson's approximation, it would merely be necessary, with four fixed centers, to use appropriately lengthened radii. The most serious defect in the shapeliness of the broken line is the abrupt discontinuity in the curvature at the four points where the radius of curvature is altered in the ratio k/l .

$$\frac{k}{l} = \frac{[\sqrt{1+x^2} - (1-x^2)]^2}{x^2(3-x^2)}.$$

(If $x < \frac{1}{2}$, $\frac{k}{l} = x^2 \left(\frac{3}{4} + \frac{x^2}{12} + \frac{107}{400}x^4 - \frac{101}{200}x^6 \right)$, nearly enough to give three

decimal places.)

When $x = \frac{1}{\sqrt{3}}$, $(E_1 = 0)$, $\frac{k}{l} = 0.268$; when $x = \frac{3}{4}$, $\frac{k}{l} = \frac{13}{27}$; when $x = \frac{1}{2}$,

$$\frac{k}{l} = 0.197.$$

For consecutive values of x , at intervals of 0.1:

$x \dots \dots \dots$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$k/l \dots \dots \dots$	0.000	0.008	0.030	0.068	0.123	0.197	0.291	0.412	0.559	0.752	1.000

In order to estimate this error on somewhat the same basis as the errors E_1, E_2, E_3 , the second of the following tables includes, under the caption $\pm E_4$, the value of

$$\frac{l - \frac{1}{2}(l + k)}{\frac{1}{2}(l + k)} = -\frac{k - \frac{1}{2}(l + k)}{\frac{1}{2}(l + k)} = \frac{1 - k/l}{1 + k/l}.$$

The first of the following tables shows certain critical values of E_1, E_2, E_3 and their derivatives; the second shows consecutive values of $E_1, E_2, E_3, \pm E_4$ and the eccentricity.

e	x	E_1	$\frac{dE_1}{dx}$	E_2	$\frac{dE_2}{dx}$	E_3	$\frac{dE_3}{dx}$
1.000	0.000	$+\infty$	$-\infty$	0.500	0.000	-1.000	2.000
0.817	0.577	0.000	-0.147	0.268	-0.642	-0.089	0.380
0.696	0.717					0.000	0.099
0.693	0.721	-0.008	0.000				
0.515	0.857					0.030	0.000
0.000	1.000	0.000	+0.035	0.000	-0.586	0.000	-0.414

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
E_1	$+\infty$	1.789	0.555	0.207	0.073	0.018	-0.004	-0.008	-0.007	-0.004	0.000
E_2	0.500	0.491	0.466	0.426	0.375	0.317	0.254	0.188	0.123	0.060	0.000
E_3	-1.000	-0.801	-0.518	-0.439	-0.243	-0.165	-0.071	-0.008	+0.025	0.027	0.000
$\pm E_4$	1.000	0.985	0.942	0.872	0.781	0.671	0.549	0.418	0.283	0.141	0.000
e	1.000	0.995	0.980	0.954	0.917	0.866	0.800	0.714	0.600	0.484	0.000

The constructions made from these data show that Mr. Iwerson's approximation is very close for a medium eccentricity, e. g. for $e = 0.92, x = 0.4$.

It is possible to calculate in advance a length to be used instead of b for $0Y$, so that the circle with center R shall pass through the end of the minor axis.

If $b/a = x, \frac{1}{2}(1-x)(\sqrt{3}-1) = p, 0Y = ya$, where

$$y^4 - (2 + p^2)y^2 + (1 - p^2) = 0;$$

or $0y = a \tan \phi$, where $\cos 2\phi \cdot \sec \phi = p$.

This might be worth while in case e is large, E_1 objectionable and E_2 and E_3 of no consequence.

As to what errors are tolerable and what objectionable, that is of course entirely a matter of circumstances.

DISCUSSIONS.

I. RELATING TO THE ORDER OF OPERATIONS IN ALGEBRA.

By N. J. LENNES, University of Montana.

§ 1. *The Rules as Given in the Books.*

Subtraction and division are defined as the inverse operations of addition and multiplication. The commutative and associative laws of addition and multiplication are, therefore, extended in the same manner to both subtraction and division. In the case of addition alone or of multiplication alone it is agreed that, when no symbols of aggregation occur, the operations are to be performed from left to right. Thus, $a + b + c$ means $(a + b) + c$, and $a \times b \times c = (a \times b) \times c$. Without such an understanding the associative laws would have no meaning.

The Commutative and Associative Laws. In case symbols of addition and subtraction both occur (and no other symbols), it is agreed that each symbol applies only to the term immediately following it, and that the operations are to be performed from left to right.

Thus, $8 - 2 + 4 = (8 - 2) + 4 = 10$, and not $8 - (2 + 4) = 2$. From this usage it follows that terms connected by + and - signs may be *commuted*, but they may not be *associated*, except when a + sign precedes the group in question. Thus, $8 + 4 - 2 = 8 + (4 - 2)$, but $8 - 4 - 2$ is not equal to $8 - (4 - 2)$.

In case the signs of multiplication and division occur with no signs of addition and subtraction intervening, and in case no symbols of aggregation are used, then it is likewise agreed (in the theoretical development in the books) that each symbol applies only to the factor (or divisor) immediately following it, and that the operations are to be performed in order from left to right.

Thus, $8 \div 2 \times 4 = (8 \div 2) \times 4 = 16$, and not $8 \div (2 \times 4) = 1$. As in the case of addition and subtraction, it results from this agreement that the commutative law applies to the operations of multiplication and division, while the associative law does not apply, except when the sign \times precedes the group in question.

That is, $8 \div 4 \times 2 = 8 \times 2 \div 4$, but $8 \div 4 \times 2$ is not equal to $8 \div (4 \times 2)$. As remarked by Chrystal, under these conventions, the associative and commutative laws for addition and subtraction are *formally identical* with these laws for multiplication and division. (*Text Book of Algebra*, Part I, page 17.)

Following this theoretical development most of the current text-books give a rule like the following:

A series of operations involving multiplication and division alone shall be performed in the order in which they occur from left to right.